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The Discrete Evasion Game

by Joseph Bram, Sept, 73

Systems Evaluation Group

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CENTER FOR NAVAL ANALYSES

RESEARCH CONTRIBUTION 244

Systems Evaluation Group

THE DISCRETE EVASION GAME

September 1973

Joseph Bram

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ABSTRACT

Theoretical and computational aspects of the three-move discrete evasion game are presented. An Evader strategy is given that yields an upper bound of .2890 for the game-value, and a Marksman strategy is given that yields a lower bound of .2842. A particular form for the Marksman strategy is presented which depends on r bits of information, and it is proved that this type of strategy is near-optimal. The results are also applied to the two-move game, which was solved earlier by other workers.

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I. INTRODUCTION

Game theory, the mathematical approach to conflict situations, has been applied to many operations research, systems analysis, and decision making problems since its development in the 1930s. Of particular significance to many military operational problems has been a class of games known as evasion games, in which one or more players attempt to evade others. The game to be discussed in this paper has just two players, a Marksman and an Evader. The Evader moves about in a grid of equally spaced points in an unbounded straight line. At each time step, say each second, he must move one unit to the right or one unit to the left. The Marksman observes the Evader's motion for as long as he likes and then fires his single weapon, which takes exactly three seconds to reach the point at which the Marksman aimed. The payoff to the Marksman is 1 if the weapon hits the Evader, i.e., if the Evader's position coincides with that of the weapon when it arrives. The payoff is 0 otherwise. The Evader cannot see the weapon in flight, and is only told after the fact that the play of the game is over.

This is a much simplified version of a class of military problems such as the following (reference (3)).

> "A ship in midocean is aware of an enemy bomber's presence, but the plane is too high for precise detection. The ship is interested only in not being hit; it has no offensive means. The plane has one bomb and we suppose -- to avoid extraneous factors -- that the bomber's aim is excellent. The ship knows this, but knows nothing about when or where the bomb will be dropped until after detonation. It is to maneuver so as to minimize the hit probability. We suppose that its only kinematic restriction is that it travels with a fixed speed v . There is a time lag T between the bomber's last sighting of the ship and detonation. Thus the bomber must aim at an anticipated position of the ship."

The problem for the idealized game is to find optimal mixed strategies for the two players, in the sense of game theory. So, if α and p denote possible strategies for the Marksman and Evader, respectively, and if $M(\alpha, p)$ is the expected payoff to the Marksman, the problem is to find α_o, p_o, so that for all allowable α ,

$$M(\alpha, p_o) \le M(\alpha_o, p_o) \quad , \qquad (1)$$

and for all allowable p ,

$$M(\alpha_o, p_o) \le M(\alpha_o, p) \quad . \qquad (2)$$

The value of the game is

$$v = M(\alpha_o, p_o)$$

and the inequalities (1) and (2) assert that when the Evader uses the strategy p_o , then no matter which α is chosen by the Marksman, his payoff will not exceed v , and when

the Marksman uses the strategy α_0 , then no matter which p is chosen by the Evader, the payoff to the Marksman will be at least v . (The payoff is the probability of hitting the Evader, which the Marksman tries to maximize, and the Evader tries to minimize.)

We do not have the solution, but we shall present a strategy for the Evader that will assure him that the Marksman will never achieve a payoff more than .289025, and a strategy for the Marksman that will assure him a payoff of .28423 . We can therefore assert that the value v satisfies

$$28423 \leq v \leq .28903 \quad . \tag{3}$$

The present game has been called the 3-move game since the Evader makes 3 moves while the weapon is in flight. The 2-move game has a corresponding definition and was practically solved in reference (1) and reference (3), but an optimal Marksman's strategy was not given until reference (2). At the end of this paper, the 2-move game is considered and the same sort of strategy is applied there as is suggested for the 3-move game.

II. A SAFE EVADER STRATEGY

The form of the game makes it clear that there is right-left symmetry; any Evader path, which is a sequence of rights and lefts, is equivalent to the same path with lefts and rights interchanged. Accordingly, each such path can be described as a sequence of straights and turns*, except for the very first move.

The best strategy that we have found for the Evader can be described in terms of the conditional probability of moving straight given the "state" S of the previous moves. A straight is denoted by 1 and a turn by 0. If the previous Evader moves to the current time n are

$$\ldots \varepsilon_{n-1}\, 0 \ ,$$

i.e., the last move was a turn, we say that the "system" (or the Evader) is in state $S=1$. If the previous moves are

$$\ldots 0\, \overbrace{1\, 1 \ldots 1}^{k-1} \ , \tag{4}$$

we say the state is $S = k$. Equivalently, the Evader is in state $S = k$ if the last k moves have been to the right (or to the left), but not the last $(k+1)$.

For the first move, the Evader can choose S to be any $k \geq 1$.

If at time n, the Evader is in state $S = k$, then at time $n+1$, the new state can be $k+1$ or 1, and nothing else. The probability p_k of moving straight is the only quantity that the Evader needs for his next move, $k=1,2,3,\ldots$. The process which consists of all possible random paths is a first order Markoff process in the states S, but is not a finite order process in terms of the previous bits (0's and 1's).

Suppose that at time n the Evader is in state k. At time $n+3$, there are exactly four places where the Evader can be, which are called $W=0,1,2,$ or 3. These positions depend on the moves Y_{n+1}, Y_{n+2}, Y_{n+3} according to the following table.

Y_{n+1}	Y_{n+2}	Y_{n+3}	W
0	0	0	1
0	0	1	2
0	1	0	1
0	1	1	0
1	0	0	2
1	0	1	1
1	1	0	2
1	1	1	3

*A straight is a left followed by a left or a right followed by a right. A turn is a right followed by a left or vice versa.

The meaning of the Ws is as follows. Suppose the Evader moved to the right at time n , and the next three moves are 0, 1, 1, i.e., turn, straight, straight. So the Evader goes left, left, left, and his position is three units back from the position at n . "Back" because he was moving to the right at n . This outcome is called W=0 . The outcome wherein the future position is one unit back is W=1; one unit ahead is W=2 , and three units ahead is W=3 . The construction of the W-table above follows from these remarks.

The Evader, who is in state k at time n , can calculate the conditional probabilities, given S = k , of the 8 possible 3-bit groups that represent his next three moves. Two of them and their probabilities are listed.

Y_{n+1}	Y_{n+2}	Y_{n+3}	Pr	W
0	0	1	$(1-p_k)(1-p_1)\, p_1$	2
1	1	0	$p_k\, p_{k+1}\, (1-p_{k+2})$	2

By going through the complete list of 8, and picking out the one for W=0 , one for W=3 , three for W=1 , and three for W=2 , we find:

$$\Pr (W=0 \mid S=k) = (1-p_k)\, p_1 p_2 \tag{5}$$

$$\Pr (W=1 \mid S=k) = (1-p_k)\, p_1\, (1-p_2) + (1-p_k)(1-p_1)^2 + p_k\, (1-p_{k+1})\, p_1 \tag{6}$$

$$\Pr (W=2) \mid S=k) = (1-p_k)(1-p_1)\, p_1 + p_k\, (1-p_{k+1})(1-p_1) + p_k\, p_{k+1}\, (1-p_{k+2}) \tag{7}$$

$$\Pr (W=3 \mid S=k) = p_k\, p_{k+1}\, p_{k+2} \quad . \tag{8}$$

These equations hold for $k=1, 2, 3, \ldots$.

Suppose the Marksman guesses $\hat{W}$ for W when S = k . Then his expected payoff will be $\Pr (W = \hat{W} \mid S = k)$. Let

$$v^* = \sup_k \sup_{\hat{W}} \Pr (W = \hat{W} \mid S = k) \quad . \tag{9}$$

Then, no matter what the Marksman does, his payoff will not exceed v^* , which depends on $p_1, p_2, p_3, \ldots$.

In table 1, a set of p_k's is presented that defines the best Evader strategy that we have been able to find. In table 2, the corresponding values of $\Pr (W = \hat{W} \mid S = k)$ are given, from the equations (5) to (8).

TABLE 1

A SAFE SET OF p_k's FOR THE EVADER

k	p_k	
1	.692900	.30710
2	.624674	.37533
3	.667745	.33226
4	.651372	.34863
5	.662413	.33759
6	.658589	.34142
7	.661352	.33864
8	.660473	.33953
9	.661162	.33884
10	.660963	.33904
11	.661136	.33887
12	.661092	.33891
13	.661135	.33886
14	.661125	.33887
15	.661136	.33886
.	.	
.	.	
.	.	

TABLE 2

$\Pr(W = \hat{W} \mid S = k)$ USING p_k's OF TABLE 1

$k \backslash \hat{W}$	0	1	2	3
1	.132924	.289025	.289025	.289025
2	.162455	.276818	.289025	.271703
3	.143812	.279046	.289025	.288177
4	.150899	.275909	.289025	.284166
5	.146120	.276335	.289025	..288520
6	.147775	.275524	.289025	.287675
7	.146579	.275596	.289025	.288799
8	.146960	.275386	.289025	.288630
9	.146662	.275394	.289025	.288919
10	.146748	.275339	.289025	.288888
11	.146673	.275339	.289025	.288963
12	.146692	.275324	.289025	.288959
13	.146673	.275323	.289025	.288978

We were led to the Evader strategy above by examining 2nd, 3rd, and 4th order processes and solving approximately for $\max_{S,W} P(W|S)$ with the aid of a nonlinear programming algorithm. The "optimal" conditional transition probabilities suggested that they depended not on all the previous bits in a 4-bit state, for example, but only on the state S as defined above. Using this statement as a hypothesis, the p's and $P(W|S)$'s of table 1 and table 2 are easily found.

III. THE FORM OF THE MARKSMAN STRATEGY

For ease of exposition, we suppose here that the Marksman uses a strategy that depends only on the Evader's most recent four straights and turns, the last four bits of information before firing. (In terms of lefts and rights, this requires five bits of information, of course.) Let straights be denoted by 1's and turns by 0's, and let

$$Y_1, Y_2, Y_3, \ldots, Y_n \ldots$$

be a sequence of 1's and 0's, i.e., a possible Evader path.

Let N be a larger integer and consider the part of the path from Y_1 to Y_N. We are concerned with the numbers n_{ij} of transitions from state i to state j, where i and j stand for any 6-bit group, from 000000 to 111111, of which there are $2^6 = 64$. By a transition, we mean an ordered pair such as

$$((Y_{15}, \ldots, Y_{20}), (Y_{16}, \ldots, Y_{21})) \quad ;$$

the first possible transition is from $(Y_1, \ldots, Y_6)$ to the next state, and the last one is from $(Y_{N-6}, \ldots, Y_{N-1})$ to the next state. So there are $N-6$ transitions under consideration. But with N very large, there are essentially N transitions.

For each of the 64 states i, there are exactly 2 possible following states j, determined by the value of the next observable y-bit. Let

$$k_i = \sum_j n_{ij}$$

and

$$m_i = \sum_j n_{ji} \quad .$$

Then k_i is the number of times there is a transition* out of state i, and m_i is the number of transitions into state i. Apart from the very first and the very last transition, every transition into state i is accompanied by a transition out of state i. It follows that for each i,

$$k_i = m_i \quad ,$$

that is,

$$\sum_j n_{ij} = \sum_j n_{ji} \quad ,$$

*A transition from i to i is to be counted also.

apart from a negligible error from the end effects. Furthermore,

$$\sum_i \sum_j n_{ij} = \sum_i k_i = N ,$$

essentially. We put

$$p_{ij} = \frac{n_{ij}}{N} .$$

So p_{ij} is the proportion of times there is a transition from i to j. We have, for each i,

$$\sum_j p_{ij} = \sum_j p_{ji} ,$$

and

$$\sum_i \sum_j p_{ij} = 1 .$$

Suppose that the Marksman fires at a time t between 4 and N-3, so that $y_1, y_2, \ldots, y_t$ have been observed. By our assumption, the Marksman can make use of the knowledge of the last 4 y's only; if $y_1, y_2, \ldots y_{t-4}$ were changed to any other y's, the behavior of the Marksman firing at time t would be unaltered. It is convenient to consider the transition from the state $i = (y_{t-3}, y_{t-2}, \ldots y_{t+2})$ to the following state $j = (y_{t-2}, y_{t-1}, \ldots, y_{t+3})$. This enables us to calculate the expected payoff to the Marksman in terms of the quantities α_{ij} and π_i, which are under the control of the Marksman and which will be described presently, and of the p_{ij}, which are under the Evader's control.

The Marksman, who is about to fire at time t, knows the last four bits, say $S = (y_{t-3}, y_{t-2}, y_{t-1}, y_t)$. There are $2^4 = 16$ such S's. The position W of the Evader at time t+3, relative to the position at t will be one of four possibilities, as a function of $y_{t+1}, y_{t+2}, y_{t+3}$. The function is given in the following table which is the same as that of section II.

y_{t+1}	y_{t+2}	y_{t+3}	W
0	0	0	1
0	0	1	2
0	1	0	1
0	1	1	0
1	0	0	2
1	0	1	1
1	1	0	2
1	1	1	3

The Marksman, firing at time t, will fire randomly at $W=0, 1, 2, 3$ with probability $\alpha_{S0}, \ldots, \alpha_{S3}$. Of course,

$$\sum_W \alpha_{SW} = 1 \quad .$$

The Marksman will require a 16x4 table of α's, the rows indexed by S, and the columns by W.

Now, we know that there is a transition from $i = (y_{t-3}, \ldots, y_{t+2})$ to $j = (y_{t-2}, \ldots, y_{t+3})$. The state i contains enough information for us to determine S, viz., the first 4 bits of i. So S can be regarded as a function of i. Also, the state j enables us to determine W; we simply look at the last 3 bits of j and then consult the W-table given above. Accordingly, W is a function of j. The result is that we can write

$$\alpha_{ij} = \alpha_{SW}$$

(with an abuse of notation that is perhaps preferable to writing $\widetilde{\alpha}_{SW}$) if and only if S is the correct value, $S(i)$, and W is the value, $W(j)$.

In the transition from i to j, the Marksman who fires here, receives 1 unit if the W he fires at coincides with $W(j)$, and 0 otherwise. His expected payoff is therefore $\alpha_{SW} = \alpha_{ij}$. If the Marksman chooses t uniformly in the large interval from 1 to N (essentially), then the transition from i to j that arises occurs with probability p_{ij}. The expected payoff to the Marksman is, accordingly,

$$M(\alpha, p) = \sum_{ij} \alpha_{ij} p_{ij} \quad .$$

But the Marksman has a better procedure, still using only the last four bits.

The correct strategy for the Marksman, assuming for definiteness that he can only use the last four bits is to choose the 16x4 α_{SW} table appropriately and also 16 probabilities π_S. His procedure is the following:

a) Choose t uniformly between 1 and N, and observe S, the last four bits.

b) With probability π_S fire. If he fires, he aims at W with probability α_{SW}, and the play is over. With probability $1 - \pi_S$, the Marksman starts over, i.e., goes back to a), resetting his clock to 0, but the y sequence now begins with y_{t+1}.

When each $\pi_S > 0$, the Marksman who follows this procedure will fire eventually with probability 1. So his payoff can be calculated as his expected return, given that he fires the first time, which is the same as when he fires the <u>kth</u> time, for any k.

The probability of a transition from i to j, when t is chosen uniformly in a large interval, is p_{ij}. The probability of firing given (i,j) is π_S where $S = S(i)$. So the probability of firing the first time is

$$\sum_{ij} \pi_i p_{ij} \, ,$$

in which we have written π_i for π_S with the understanding that

$$\pi_i = \pi_S$$

if and only if $S = S(i)$, the first four bits of i. The payoff, given a transition from i to j and a firing, is α_{ij}. The expected payoff to the Marksman under this policy is therefore

$$M(\alpha, \pi; p) = \frac{\sum_{ij} \alpha_{ij} \pi_i p_{ij}}{\sum_{ij} \pi_i p_{ij}} \quad . \tag{9}$$

The solution of the game with this payoff is discussed in section IV. We will prove that there is a number v^*, and a near-optimal strategy α^*_{SW} and π^*_S for the Marksman, and a strategy p^*_{ij} for the Evader with the following properties:

1) If the Evader uses p^*_{ij}, then no strategy for the Marksman that uses only the last 4 bits can yield him a payoff exceeding v^*. It is not required that the Marksman restrict himself to a policy of the type that leads to equation (9) above. So long as the information available to the Marksman is restricted to the last 4 bits, he cannot achieve more than v^*.

2) If the Marksman uses the policy described above with α^*_{ij} and π^*_i, then no matter what strategy is followed by the Evader, pure or mixed, the Marksman will obtain a payoff equal to or greater than $v^* - 10^{-10}$, say.

The value v^* and the strategies (α^*, π^*) and p^* depend on the number of bits available to the Marksman. We have been talking about 4 bits, but evidently everything in this section can be reread with 5 bits or r bits instead of 4. If the Marksman can use r bits, the appropriate strategy for the Evader will depend on $r+2$ bits. If v^*_r denotes the value when r bits are used by the Marksman, we have clearly, $v^*_r \le v^*_{r+1}$ for every r, and the value of the original game is

$$\tilde{v} = \lim_{r \to \infty} v^*_r \quad .$$

We have been unable to obtain this limit. The best we can say with certainty is that

$$.28423 \leq \tilde{v} \leq .28902 \quad . \tag{10}$$

Let us note that when the Evader must consider (r+2) - bit groups, the number of quantities p_{ij} that must be found is twice the number of i - states, i.e., $2x2^{r+2} = 2^{r+3}$. For r=4, this means 128 p's, and for r=5, we would need 256 p's. In the latter case, we would require the solution of a few linear programming problems with tableaus of 257 rows and 258 columns, which we declined to pursue. Our lower bound of .28423 is the assured payoff to the Marksman when he uses a strategy which contains an amount of information intermediate to 4 bits and 5 bits and will be discussed below.

IV. CALCULATING A NEAR-OPTIMAL STRATEGY

We start here with the payoff equation (9) of the last section,

$$M(\alpha, \pi; p) = \frac{\sum_{ij} \alpha_{ij} \pi_i p_{ij}}{\sum_{ij} \pi_i p_{ij}} .$$

For definiteness, the reader may think of i as a 6-bit group, S(i) is determined by the first 4 bits of i , etc., as in the last section. The $\alpha_{ij} = \alpha_{SW}$ are non-negative with

$$\sum_W \alpha_{SW} = 1$$

for each S . The $\pi_i = \pi_S$ are between 0 and 1. The p_{ij} are non-negative with

$$\sum_j p_{ij} = \sum_j p_{ji}$$

for each i , and $\sum_{ij} p_{ij} = 1$. We will show here how this "4-bit" game can be solved; in principle the "r-bit" game goes the same way.

Let p_{ij} be an allowable set of p's for the Evader. If i denotes a 6-bit group, then p_{ij} is precisely the probability of a 7-bit group, whose first 6 bits correspond to i . Let S_o be any 4-bit group. Then the probability that S_o arises can be obtained by adjoining 3 more bits to S_o in the 8 possible ways, to get eight 7-bit groups and then summing over the 8 appropriate p_{ij}'s. We have

$$Pr(S_o) = \sum \{p_{ij} : S(i) = S_o\} ,$$

a sum of 8 terms. If W_o is a possible value of $W = W(j)$, then the probability that in all possible 7-bit groups, we have $S(i) = S_o$ and $W(j) = W_o$, is simply the sum of all the p_{ij} such that $S(i) = S_o$ and $W(j) = W_o$. Such a sum will contain 1 term for $W_o = 0$ or $W_o = 3$; it will contain 3 terms if $W_o = 1$ or $W_o = 2$, as can be seen from the W-table of the last section.

Consider the numerator in equation (9). Since $\alpha_{ij} = \alpha_{SW}$ and $\pi_i = \pi_S$, it is convenient to hold S, W fixed, then sum over p_{ij} to get Pr(S, W), etc. We obtain

$$\sum_{ij} \alpha_{ij} \pi_i p_{ij} = \sum_{SW} \alpha_{SW} \pi_S \Pr(S,W) = \sum_S \pi_S \sum_W \alpha_{SW} \Pr(W|S) \Pr(S) .$$

Since

$$\sum_W \alpha_{SW} = 1 \text{ for each } S ,$$

we have

$$\sum_W \alpha_{SW} P(W|S) \le \max_W \Pr(W|S) .$$

Therefore,

$$\sum_{ij} \alpha_{ij} \pi_i p_{ij} \le \sum_S \pi_S \Pr(S) \max_W \Pr(W|S) .$$

For the denominator in equation (9), we have

$$\sum_{ij} \pi_i p_{ij} = \sum_S \pi_S \Pr(S) .$$

We conclude that

$$M(\alpha, \pi; p) \le \frac{\sum_S \pi_S \Pr(S) \max_W \Pr(W|S)}{\sum_S \pi_S \Pr(S)}$$

and finally,

$$M(\alpha, \pi; p) \le \max_S \max_W \Pr(W|S), \tag{11}$$

valid for all allowable α_{ij}, π_i, p_{ij}.

The r.h.s. of equation (11) depends on the p_{ij} but not the α's or π's.

It is also clear from the relations preceding equation (11) that, if p_{ij} were announced to the Marksman, he could achieve the payoff

$$\max_{S,W} \Pr(W|S) \tag{12}$$

by properly choosing α_{ij} and π_i . The conclusion we draw is that the correct strategy for the Evader in our present context is to choose p_{ij} so that the number (12) is smallest possible. These remarks provide the motivation for the following considerations.

Let $p = \{p_{ij}\}$ denote the vector with components p_{ij}, wherein the (i,j) pairs -- 128 of them in the present case -- have been indexed from 1 to 128 in any convenient manner. Let v be an arbitrary real number between 0 and 1 . For each v , define

$$D(v) = \Big\{p : \text{all the } p_{ij} \geq 0 ,$$

$$\forall\, i : \sum_j p_{ij} - \sum_j p_{ji} \leq 0 ,$$

$$\sum_{ij} p_{ij} = 1 \quad ,$$

$$\forall\, S_o, W_o : \sum_{\substack{S(i)=S_o \\ W(j)=W_o}} p_{ij} - v \sum_{S(i)=S_o} p_{ij} \leq 0 \Big\} \quad .$$

For each v, $D(v)$ is a compact polyhedral set, i.e., a set defined by linear inequalities.

(Note: For any p , if $p \in D(v)$, then

$$\forall\, i : \sum_j p_{ij} - \sum_j p_{ji} = 0 \quad ,$$

which follows by summing over i .) We have $D(0) = 0$, the empty set, and $D(1) \neq 0$. Also $D(v)$ is increasing with v . It follows that there is a v^* such that $D(v) \neq 0$ if $v > v^*$ and $D(v) = 0$ if $v < v^*$. Let $v_n = v^* + 2^{-n}$, $n=1,2,3,\ldots$. The sets $D(v_n)$ form a decreasing sequence of non-empty compact sets, and there is, accordingly, a point p^* that belongs to all of the $D(v_n)$. For such a point p^* , we have

$$\forall\, S_o, W_o: \sum_{\substack{S(i)=S_o \\ W(j)=W_o}} p^*_{ij} - v_n \sum_{S(i)=S_o} p^*_{ij} \le 0$$

for each n, and therefore we can take the limit as $n \to \infty$. The result is that $p^* \in D(v^*)$, as we would expect.

We are ready to prove one half of the claim following equation (9) of the last section. The number v^* and the Evader strategy $\{p^*_{ij}\} = p^*$ are as described in the previous paragraph. All that is needed is to show that when the Evader uses p^*, and the Marksman is restricted to the information contained in any observable S_o, the Marksman cannot achieve more than v^*. But this follows since $p^* \in D(v^*)$, so that

$$\forall\, S_o, W_o: \ \frac{\Pr(S_o, W_o)}{\Pr(S_o)} \le v^*, \text{ i.e., } \Pr(W_o \mid S_o) \le v^* \tag{13}$$

for every W_o and every S_o for which $\Pr(S_o) \neq 0$. If $\Pr(S_o) = 0$ for some S_o, the Marksman will never observe S_o, and there's no problem. (In the next section, we show that $P(S_o) > 0$ if $p \in D(v^*)$. If the Marksman observes S_o and fires, his payoff cannot exceed v^* by equation (13).

We turn now to the problem of calculating v^* and near-optimal α_{ij} and π_i for the Marksman. To calculate v^*, or a close approximation, we need only form some objective function $\sum_{ij} \epsilon_{ij} p_{ij}$ with $\epsilon_{ij} \ge 0$ and minimize this function subject to the constraints given above in the definition of $D(v)$, for fixed v. The LP (linear programming) algorithm yields an "optimal" p in case $D(v) \neq 0$, and gives another indication when $D(v) = 0$. So, for example, we can take $A=.25$ and $B=.30$, take the midpoint as v, and use LP. If $D(v) = 0$, replace A by v; if $D(v) \neq 0$, replace B by v; then do it again. After 8 or 9 repetitions, the interval (A,B) is very small and we have a good approximation to v^*.

Now we shall modify the previous LP problems slightly, to get α^*_{ij} and π^*_i. Let

$$g(p) = \sum_{ij} \epsilon_{ij} p_{ij}$$

be the function to be minimized where $\epsilon_{ij} \ge 0$, and, in addition, ϵ_{ij} depends only on S and W. So $\epsilon_{ij} = \epsilon_{k\ell}$ if and only if $S(i) = S(k)$ and $W(j) = W(\ell)$. Define

$$E_S = \sum_W \epsilon_{SW} \quad .$$

Let $v \geq v^*$. The constraints we shall use here are, besides $p_{ij} \geq 0$:

$$\forall\ i: \sum_j p_{ij} - \sum_j p_{ji} \leq 0$$

$$\sum_{ij} p_{ij}\ E_i \geq 1$$

where $E_i = E_S$ when the first four bits of i comprise S . Also,

$$\forall\ S_o, W_o: \sum_{\substack{S(i)=S_o \\ W(j)=W_o}} p_{ij} - v \sum_{S(i)=S_o} p_{ij} \leq 0 \quad .$$

Since $v \geq v^*$, the solution of this LP problem exists. By the duality theorem, the dual problem has a solution, and the value of the maximum in the dual is the value of the minimum in the primal problem.

The dual problem is obtainable as follows. We introduce the Lagrange multipliers φ_i, κ , λ_{SW} corresponding to the primal constraints, and form the Lagrangian function:

$$L = \sum_{ij} \epsilon_{ij}\ p_{ij} - \sum_i \varphi_i \Big(\sum_j p_{ji} - \sum_j p_{ij}\Big)$$

$$- \kappa \Big(\sum_{ij} p_{ij}\ E_i - 1\Big)$$

$$- \sum_{S_o W_o} \lambda_{S_o W_o} \Big(v \sum_{S(i)=S_o} p_{ij} - \sum_{\substack{S(i)=S_o \\ W(j)=W_o}} p_{ij}\Big) \quad .$$

We rewrite this by collecting together all the p_{ij} terms:

$$L = \sum_{ij} p_{ij} \Big(\epsilon_{ij} - \varphi_j + \varphi_i - \kappa\ E_i - v\ \ \Lambda_i + \lambda_{ij}\Big) + \kappa \quad ,$$

where

$$\Lambda_i = \sum_W \lambda_{S_o W}$$

if and only if

$S(i) = S_o$ and $W(j) = W_o$. The dual problem is to find

$$\max_{\varphi, \kappa, \lambda} \quad (\kappa)$$

subject to the constraints, besides $\varphi \geq 0$, $\kappa \geq 0$, $\lambda \geq 0$:

$$\forall\, ij: \epsilon_{ij} - \varphi_j + \varphi_i - \kappa\, E_i - v\, \Lambda_i + \lambda_{ij} \geq 0 \ .$$

Let φ^* , κ^* , λ^* denote a solution to this problem. (These values are available as soon as the primal LP problem is solved.)

Now let p_{ij} be <u>any</u> allowable set of p's for the Evader in our context. The only constraints are: $p_{ij} \geq 0$, $\sum_j p_{ij} = \sum_j p_{ji}$, and $\sum_{ij} p_{ij} = 1$. Multiply each dual constraint above by p_{ij} and sum. We get

$$\sum_{ij} \epsilon_{ij}\, p_{ij} - \kappa^* \sum_{ij} E_i\, p_{ij} - v \sum_{ij} \Lambda^*_i\, p_{ij} + \sum_{ij} \lambda^*_{ij}\, p_{ij} \geq 0$$

because the terms involving φ^*_i cancel.

Let σ be the smaller of κ^* and v . Then we obtain from the last relation:

$$\sum_{ij} p_{ij}\, (\epsilon_{ij} + \lambda^*_{ij}) \geq \sigma \sum_{ij} p_{ij}\, (E_i + \Lambda^*_i) \quad ,$$

valid for every allowable p_{ij} . Finally, if for each i (for each S, really), $E_i + \Lambda^*_i > 0$, we can divide and get:

$$\frac{\sum_{ij} p_{ij}\, (\epsilon_{ij} + \lambda^*_{ij})}{\sum_{ij} p_{ij}\, (E_i + \Lambda^*_i)} \geq \sigma \qquad (14)$$

for every allowable p_{ij} . The last few paragraphs may be summarized thus:

Let ϵ_{ij} be the non-negative coefficients of the "objective" function $\sum_{ij} \epsilon_{ij}\, p_{ij}$ and let ϵ_{ij} depend only on S and W. Let $E_S = \sum_W \epsilon_{SW}$ and replace the constraint $\sum p_{ij} = 1$ by $\sum_{ij} p_{ij}\, E_i \geq 1$ in the definition of D(v). Let $v \geq v^*$. Solve the LP problem of minimizing $\sum \epsilon_{ij}\, p_{ij}$, and let κ^* be the value of the minimum. Let λ^*_{SW} be the appropriate dual variables, and put σ = smaller of κ^* and v, and let $\Lambda^*_S = \sum \lambda^*_{SW}$. Then equation (14) holds for every allowable p_{ij}, provided that $E_S + \Lambda_S > 0$ for each S.

At this point, we can set

$$\alpha_{SW} = \frac{\epsilon_{SW} + \lambda^*_{SW}}{E_S + \Lambda^*_S}$$

for each S, W, and

$$\pi_S = \frac{E_S + \Lambda^*_S}{\max\limits_{S_o} (E_{S_o} + \Lambda^*_{S_o})} \quad .$$

Then $\sum_W \alpha_{SW} = 1$, and each $\pi_S \leq 1$,

and equation (14) becomes

$$\frac{\sum_{ij} p_{ij}\, \alpha_{ij}\, \pi_i}{\sum_{ij} p_{ij}\, \pi_i} \geq \sigma$$

for all allowable p_{ij}. Comparing this with equation (9), we see that α_{ij} and π_i define a strategy for the Marksman that will assure him the payoff σ. (Therefore, $\sigma \leq v^*$, so that $\sigma = \min(\kappa^*, v) = \kappa^*$.) These remarks suggest an iterative computation procedure for getting better and better α's and π's, which was found to be extremely effective in getting near optimal strategies.

V. EXISTENCE PROOF FOR NEAR-OPTIMAL STRATEGIES

To prove the existence of a pair α^*, π^* that will guarantee a payoff of $v^* - 10^{-10}$, we proceed as follows. We consider, in a manner somewhat different from previously, the non-linear programming problem of finding an allowable $\{p_{ij}\}$ for which $\max_S \max_W P(W|S)$ is as small as possible. For this purpose, we define

$$E = \{< p, v >: p \in D(v) \}$$

where $D(v)$ is as defined above, and we put

$$f(p,v) = -v$$

for $(p,v) \in E$. The problem is to get $\max_{(p,v)\in E} f$. Since E is compact, the maximum exists. If (p^*, v^*) is a point of E where the maximum occurs, we see that v^* is the same as our previous v^*. The present considerations, however, enable us to invoke the Kuhn-Tucker theorem. We note that the constraints on (p,v) are precisely those of the definition of $D(v)$, except that v is now regarded as variable. The Kuhn-Tucker theorem says that, provided E satisfies a mild regularity condition, which we suppose to be true, there are Lagrange multipliers μ_{ij}, φ_i, σ, λ_{SW}, all ≥ 0 (except possibly σ), such that at the point (p^*, v^*) in E, the Lagrangian

$$J = -v + \sum_{ij}\mu_{ij}\, p_{ij} - \sum_i \varphi_i\Big(\sum_j p_{ij} - \sum_j p_{ji}\Big) - \sigma\Big(\sum_{ij} p_{ij} - 1\Big)$$
$$- \sum_{S_oW_o} \lambda_{S_oW_o}\Bigg(\sum_{\substack{S(i)=S_o\\ W(j)=W_o}} p_{ij} - v \sum_{S(i)=S_o} p_{ij}\Bigg)$$

has a zero gradient, i.e., $\partial J/\partial p_{ij} = 0$ for every (i,j), and $\partial J/\partial v = 0$. We form all these derivatives and conclude that

$$\forall\,(i,j) : \sigma + \varphi_i - \varphi_j + \lambda_{ij} - v^* \Lambda_i \geq 0$$

and

$$\sum_{SW} \lambda_{SW}\, p^*(S) = 1 \quad ,$$

where

$$\Lambda_i = \Lambda_S = \sum_W \lambda_{SW} \quad .$$

Furthermore, $\sigma = 0$ because the constraint $\sum_{ij} p_{ij} = 1$ could be replaced by $\sum_{ij} p_{ij} = 2$ or any positive constant without changing v^*. As we did earlier, we can now multiply each of the (i, j) inequalities by p_{ij} and obtain

$$\forall\, p_{ij} : \sum_{ij} p_{ij}\, \lambda_{ij} - v^* \sum_{ij} \Lambda_i\, p_{ij} \geq 0 \tag{15}$$

provided $\sum_j p_{ij} = \sum_j p_{ji}$ for each i. We know that not all the Λ's are zero because of our relation $\sum_S \Lambda_S\, p^*(S) = 1$, above. However, it is still conceivable that for some allowable $\{p_{ij}\}$, $\sum_{ij} \Lambda_i\, p_{ij} = 0$, in which case we cannot be satisfied with our λ_{SW}.

In this event, we restrict attention temporarily to $D(v^*)$. We can observe quickly that if $p \in D(v^*)$, then for every S, $P(S) = \sum_{S(i)=S} p_{ij} > 0$. For if $P(S) = 0$ for a four-bit group S, e.g., $S = 1101$, then the state $\hat{S} = y\,110$ will have some $P(\hat{S} \mid W) \geq 1/3 > v^*$ because $W=3$ cannot follow $\hat{S}$. (If $S = 1100$, then $W=0$ cannot follow $\hat{S}$.) But $P(\hat{S} \mid W) > v^*$ is impossible for $p \in D(v^*)$, and therefore $P(y\,110) = 0$ for $y = 1$ or 0, i.e., $P(110) = 0$. Similarly $P(11) = 0$, and $P(1) = 0$. Therefore all the $p_{ij} = 0$ except for $i = (000000)$, and clearly, $p \notin D(v^*)$, a contradiction.

So if $p \in D(v^*)$, we have $\sum_{ij} \Lambda_i\, p_{ij} > 0$, and since $D(v^*)$ is compact, the smallest such sum exists and is strictly positive. We have for p in $D(v^*)$:

$$\frac{\sum_{ij} \lambda_{ij}\, p_{ij}}{\sum_{ij} \Lambda_i\, p_{ij}} \geq v^* \quad ,$$

by equation (15) and therefore also, equality. We can now change the λ_{ij} to $\lambda_{ij} + \epsilon$ with $\epsilon > 0$ and small. The corresponding quotient is, for p in $D(v^*)$,

$$\frac{\sum_{ij} (\lambda_{ij} + \epsilon)\, p_{ij}}{\sum_{ij} (\Lambda_i + 4\,\epsilon)\, p_{ij}} = \frac{v^* \sum_{ij} \Lambda_i\, p_{ij} + \epsilon}{\sum_{ij} \Lambda_i\, p_{ij} + 4\,\epsilon} \quad ,$$

and with $\sum \Lambda_i\, p_{ij}$ bounded away from zero, we have shown: For every $\eta > 0$, there is

a "vector" $\{\lambda_{SW}\}$ with strictly positive components such that for every p in $D(v^*)$,

$$\frac{\sum_{ij} \lambda_{ij} \, p_{ij}}{\sum_{ij} \lambda_i \, p_{ij}} > v^* - \eta \quad .$$

Finally, this $\{\lambda_{SW}\}$ can be converted to the new one $\{\lambda_{SW} + \lambda^*_{SW}\}$ by the procedure described following equation (14). We then obtain α_{SW} and π_S such that for every allowable p , we have

$$\frac{\sum_{ij} \pi_i \, \alpha_{ij} \, p_{ij}}{\sum_{ij} \pi_i \, p_{ij}} > v^* - \eta \quad .$$

This completes the proof of the existence of near-optimal strategies (α^*, π^*) for the Marksman in the 4-bit game, and the same proof holds for every r-bit game.

VI. THE 4-1/2 BIT GAME

The values v^* for the 3-bit game and 4-bit game are about .2812 and .28395, respectively. Since Matula, reference (4), had already proved the existence of near-optimal strategies for the Marksman with payoffs arbitrarily close to $23/81 = .28395$, and the 5-bit game appeared too large for the computer, it seemed that our procedures were doomed to be dominated by Matula's strategy. However, for the 2-move game, our procedure yielded a payoff of .38193 (the value of that game is known; $v=1/2\,(3-\sqrt{5}) = .38197$), while the corresponding Matula strategy yields $3/8 = .375$, which appeared to promise some hope for our procedure.

More important was the observation that the Marksman strategy in the 2-move game depended not on all the information in the last 4-bits, for example, but only on the number k of consecutive 0's (turns) to the most recent bit, i.e., the appropriate state S is defined to be $S = k$ if the last k bits are "0", but not the last $(k+1)$, $k=0$ or 1 or .... This type of state was not useful for the 3-move game, but the following was.

The Marksman uses the state $S=(\tilde{S}, \epsilon, \epsilon')$, defined thus: ϵ' is the last observed bit, ϵ is the next to last; $\tilde{S} \epsilon \{0, 1, 2, 3\}$, $\tilde{S}$ is the largest number ≤ 3 of consecutive 0's at the end after deletion of the 2 bits ϵ, ϵ'. For example, if the last 5-bits are 00010, the state is (3, 1, 0); for 11001, the state is (1, 0, 1). There are 16 such states S. The Marksman strategy (α_{SW} and π_S) depends on these 16 states, which contain less information than 5-bits. We call the game with this kind of Marksman strategy, the 4-1/2 bit game.

The Evader's answering strategy is a set of p_{ij}'s, in which each i is an ordered triple $(S, \epsilon'', \epsilon''')$ where S is a "Marksman state," of which there are 16, and ϵ'', ϵ''' are the next 2 bits of the Evader path. So the set of Evader states contains 64 elements, and there are 128 possible transition pairs (i, j). Everything in the last three sections applies to the present game with little or no change. In the calculations, the only part that requires changes is the indexing of the (i, j)'s, and the handling of the constraints

$$\sum_j p_{ij} - \sum_j p_{ji} \leq 0, \text{ for each } i.$$

We found that v^* for this game was .28426 with a possible error in the 5-th place of a couple units. More to the point is the following near-optimal strategy, which guarantees the Marksman the expectation .284227.

In table 3a, the 16 π's are given in the order of the states S; the first four states are (000), (001), (010), (011); the last four states are (300), (301), (310), (311). If $S = (1, 0, 1)$, then $\pi_S = .382898$. In table 3b, the 16 rows correspond to the states S and the four columns correspond to $W=0, 1, 2, 3$. The Marksman never fires at $W=0$.

TABLE 3a

NEAR-OPTIMAL π_S

.457108	.141443	.536567	.141443
.480058	.382898	.536567	.141443
.157777	.254466	1.000000	.141443
.138309	.371974	.670319	.141443

TABLE 3b

NEAR-OPTIMAL α_{SW}

0.000000	0.149888	0.458426	0.391686
0.000000	0.215687	0.500000	0.284313
0.000000	0.302321	0.377268	0.320411
0.000000	0.215687	0.500000	0.284313
0.000000	0.271635	0.395957	0.332408
0.000000	0.289874	0.394900	0.315225
0.000000	0.302321	0.377268	0.320411
0.000000	0.215687	0.500000	0.284313
0.000000	0.310865	0.627440	0.061695
0.000000	0.267940	0.425974	0.306086
0.000000	0.263036	0.383736	0.353228
0.000000	0.215687	0.500000	0.284313
0.000000	0.354622	0.645378	0.000000
0.000000	0.183297	0.449359	0.367344
0.000000	0.271040	0.387236	0.341724
0.000000	0.215687	0.500000	0.284313

The guaranteed payoff of .28423 that the Marksman can depend on when he uses the strategy given above is the precise value that arises when the Evader uses, for example, the periodic path

... 100000 ...

with period 6 (1 = straight, 0 = turn). We can verify this as follows. There are 6 states S that the Marksman can observe and they arise with equal probability. Starting at the point where 5 turns are observed, the states and corresponding Ws are:

	(3,0,0)	2
	(3,0,1)	1
	(3,1,0)	1
	(0,0,0)	1
	(1,0,0)	2
and	(2,0,0)	1

From tables 3a and 3b, the corresponding α's and π's are:

α_{SW}	π_S
.645378	.138309
.183297	.371974
.271040	.670319
.149888	.457108
.395957	.480058
.310865	.157777

The weighted average of these α's with the weights π is .28423 .

The reader may wish to test the payoff of .28423 by checking, in a similar manner, the results when the Evader uses other periodic paths.

VII. THE TWO-MOVE GAME

The theory of the r-bit game (wherein the Marksman makes use only of the most recent r observed straights or turns) carries over with only few changes for the 2-move game. If the Marksman state is S , then the Evader state is $i = (S, \epsilon)$, where ϵ is the bit following the last bit observed by the Marksman. So, if S consists of 4-bits, the Evader state i is 5-bits, and the (i, j) transition is defined by 6-bits. The W-table for this game is:

ϵ	ϵ'	W
0	0	1
0	1	0
1	0	1
1	1	2

We found that the near-optimal Marksman strategy, α_{SW} and π_S , depended only on the number k, where k = largest integer m such that "the last m bits of the observed Evader path are 0" is valid. So we then took the typical Marksman state S to be such a k , with $0 \le k \le 15$. We interpret S = 15 to mean that there are 15 or more 0's in the most recent Evader path history. Again, the typical Evader state has the form $i = (S, \epsilon)$.

The best Marksman strategy we found here guarantees the Marksman a payoff of .381934 . The true value of the game is $v = 1/2(3 - \sqrt{5}) = .381966$. We present the π_S and α_{SW} in tables 4a and 4b. The first π_S is for S = 0 and the last is for S = 15 . The first row of the α table is for S = 0 , etc.

TABLE 4a

NEAR-OPTIMAL π_S

.002507	.005603	.010605	.018675
.031672	.052535	.085848	.138572
.220782	.345664	.526395	.762962
.999554	1.000000	1.000000	1.000000

TABLE 4b

NEAR-OPTIMAL α_{SW}

0.000000	0.618061	0.381939
0.000000	0.552826	0.447174
0.000000	0.527910	0.472090
0.000000	0.515848	0.484152
0.000000	0.509345	0.490655
0.000000	0.505634	0.494366
0.000000	0.503448	0.496552
0.000000	0.502136	0.497864
0.000000	0.501341	0.498659
0.000000	0.500856	0.499144
0.000000	0.500562	0.499438
0.000000	0.500388	0.499612
0.000000	0.500296	0.499704
0.000000	0.500296	0.499704
0.000000	0.500000	0.500000
0.000000	0.500000	0.500000

As an example, let us apply this strategy to the following Evader path, which is periodic with period 5 (1 = straight, 0 = turn):

... 10010 10010 ...

The states S, which occur with equal probability, and the corresponding Ws are:

S	W	$M = \alpha_{SW}$	$P = \pi_S$
0	1	.618061	.002507
1	0	0	.005603
2	1	.527910	.010605
0	0	0	.002507
1	1	.552826	.005603

The weighted average of the Ms with weights P is .381936 .

REFERENCES

(1) Dubins, L. E., "A Discrete Evasion Game " Inst. Air Weapons Research Tech. Note 2

(2) Ferguson, T. S., "On Discrete Evasion Games with a Two-Move Information Lag," Proc. of the Fifth Berkeley Symposium on Math. Stat. and Prob., Vol. 1, Univ. of Calif. Press, 1967

(3) Isaacs, R., "A Game of Aiming and Evasion: General Discussion and the Marksman's Strategies," RAND Memo RM-1385, 1954

(4) Matula, D., "Games of Sequence Prediction," O. R. Center, Univ. of Calif. Berkeley, ORC 66-3, Feb 1966

(5) Washburn, A. R., "An Introduction to Evasion Games," Naval Postgrad. School, Monterey, Calif. NPS55WS71091A, 1971

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13. ABSTRACT

Theoretical and computational aspects of the three-move discrete evasion game are presented. An Evader strategy is given that yields an upper bound of .2890 for the game-value, and a Marksman strategy is given that yields a lower bound of .2842 . A particular form for the Marksman strategy is presented which depends on r bits of information, and it is proved that this type of strategy is near-optimal. The results are also applied to the two-move game, which was solved earlier by other workers.

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